Maximum Renyi entropy principle for systems with power–law Hamiltonians

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Abstract

The Renyi distribution ensuring the maximum of a Renyi entropy is investigated for a particular case of a power–law Hamiltonian. Both Lagrange parameters, α and β can be excluded. It is found that β does not depend on a Renyi parameter q and can be expressed in terms of an exponent κ of the power–law Hamiltonian and an average energy U. The Renyi entropy for the resulted Renyi distribution reaches its maximal value at $q = 1/(1+\kappa)$ that can be considered as the most probable value of q when we have no additional information on behaviour of the stochastic process. The Renyi distribution for such q becomes a power–law distribution with the exponent $-(\kappa+1)$. When $q = 1/(1+\kappa)+\epsilon$ ($0 < \epsilon \ll 1$) there appears a horizontal "head" part of the Renyi distribution that precedes the power–law part. Such a picture corresponds to observables.

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1 Introduction

Numerous examples of power–law distributions (PLD) are well-known in different fields of science and human activity [1]. Power laws are considered [2] as one of signatures of complex self-organizing systems. They are sometimes called Zipf–Pareto law or fractal distributions. We can mention here the Zipf–Pareto law in linguistics [3], economy [4] and in the science of sciences [5], Gutenberg-Richter law in geophysics [6], PLD in critical phenomena [7], PLD of avalanche sizes in sandpile model for granulated media [8] and fragment masses in the impact fragmentation [9, 10], etc.

Graphically, PLD is presented by a linear graph in a double logarithmic plot of frequency or cumulative number as a function of size. It should be noticed here that, in general, double logarithmic plots of data from phenomena in nature or economy often exhibit a limited linear regime preceded by a near-horizontal "head" part and followed by a tail of significant curvature. The latter deviation from a power–law description can be explained by a finite-size effect. Really, for instance for the impact fragmentation, extrapolation of the PLD to infinite fragment masses would predict masses surpassing a mass of the target. This effect will not be considered here.

Successful derivations of PLD with the head part are based on Renyi or Tsallis distributions ensuring maximums of Renyi or Tsallis entropies correspondingly (see,e. g. [11, 12]). However, some parameters (q -parameter, Lagrange multiplier β) were left to be indeterminate there.

Here, a derivation of PLD will be performed on the base of maximum entropy principle (MEP) for the Renyi entropy. In the special case that a Hamiltonian of the system is a power–law function of a variable of the system the Lagrange multiplier β will be excluded at all from the Renyi distribution (Sec. 2) and the q -parameter will be determined uniquely with the use of expansion of the MEP (Sec. 3).

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According to the well-known maximum entropy principle (MEP) developed by Jaynes [13] for a Boltzmann-Gibbs statistics an equilibrium distribution of probabilities $p = \{p_i\}$ must provide maximum of the Boltzmann information entropy S_B upon additional conditions of normalization $\sum_i p_i = 1$ and a fixed average energy $U = \langle H \rangle_p \equiv \sum_i H_i p_i$.

Then, the distribution $\{p_i\}$ is determined from the extremum of the functional

$$L_G(p) = -\sum_i^W p_i \ln p_i - \alpha_0 \sum_i^W p_i - \beta_0 \sum_i^W H_i p_i, \tag{1}$$

where α_0 and β_0 are Lagrange multipliers. Its extremum is ensured by the Gibbs canonical distribution, in which $\beta_0 = 1/k_B T$ and T is the thermodynamic temperature, so there is no necessity to invoke the average energy U for exclusion of the Lagrange parameter β_0 from the Gibbs distribution because of a well-established correspondence between Gibbs thermostatistics and classical thermodynamics. On contrary, in applications of MEP to the Renyi entropy, the Lagrange parameter β may depend on the q-parameter and its physical meaning is not evident, so it is necessary to exclude it with the use of the additional condition of MEP related to the fixed average energy.

2 MEP for the Renyi entropy

If the Renyi entropy

$$S_R = \frac{k_B}{1-q} \ln \sum_i p_i^q \tag{2}$$

is used instead of the Boltzmann entropy the equilibrium distribution must ensure maximum of the functional

$$L_R(p) = \frac{1}{1-q} \ln \sum_i^W p_i^q - \alpha \sum_i^W p_i - \beta \sum_i^W H_i p_i, \tag{3}$$

where α and β are Lagrange multipliers. Note that $L_R(p)$ passes to $L_G(p)$ in the $q \to 1$ limit.

We equate a functional derivative of $L_R(p)$ to zero, then

$$\frac{\delta L_R(p)}{\delta p_i} = \frac{q}{1-q} \frac{p_i^{q-1}}{\sum_j p_j^q} - \alpha - \beta H_i = 0. \tag{4}$$

To exclude the parameter α we can multiply this equation by p_i and sum up over i, with account of normalization condition $\sum_i p_i = 1$. Then we get

$$\alpha = \frac{q}{1-q} - \beta U \tag{5}$$

and

$$p_i = \left(\sum_j^W p_j^q \, (1 - \beta \frac{q-1}{q} \Delta H_i) \right)^{\frac{1}{q-1}}, \quad \Delta H_i = H_i - U. \tag{6}$$

Using once more the condition $\sum_i p_i = 1$ we get

$$\sum_j^W p_j^q = \left(\sum_i^W (1 - \beta \frac{q-1}{q} \Delta H_i)^{\frac{1}{q-1}} \right)^{-(q-1)}$$

and, finally,

$$p_i = p_i^{(R)} = Z_R^{-1}\left(1-\beta\frac{q-1}{q}\Delta H_i\right)^{\frac{1}{q-1}} \tag{7}$$

$$Z_R^{-1} = \sum_i \left(1-\beta\frac{q-1}{q}\Delta H_i\right)^{\frac{1}{q-1}}. \tag{8}$$

This distribution satisfies to MEP for the Renyi entropy and may be called the Renyi distribution. When $q \to 1$ the distribution $\{p_i^{(R)}\}$ becomes the Gibbs canonical distribution. In this limit $\beta/q \to \beta_0 = 1/k_BT$. Such behaviour is not enough for unique determination of β. Really, in general it may be an arbitrary function $\beta(\beta_0, q)$ which becomes β_0 in the limit $q \to 1$.

To find an explicit form of β we return to the additional condition of pre-assigned average energy $U = \sum_i H_i p_i$ and substitute there the Renyi distribution (7). For definiteness sake, we will confine the discussion to the particular case of a power-law dependence of the Hamiltonian on a parameter x

$$H_i = Cx_i^\kappa. \tag{9}$$

If the distribution $\{p_i\}$ allows for smoothing over the range much larger an average distance $\Delta x_i = x_i - x_{i+1}$ without sufficient loss of information, we can pass from the discrete variable x_i to the continuous one x. Then the condition of a fixed average energy becomes

$$Z^{-1}\int_0^\infty Cx^\kappa\left(1-\beta\frac{q-1}{q}(Cx^\kappa - U)\right)^{\frac{1}{q-1}}dx = U, \quad H(x) = Cx^\kappa \tag{10}$$

or

$$Z^{-1}\int_0^\infty C_u x^\kappa\left(1-\beta U\frac{q-1}{q}(C_u x^\kappa - 1)\right)^{\frac{1}{q-1}}dx = 1, \tag{11}$$

$$Z = \int_0^\infty \left(1-\beta U\frac{q-1}{q}(C_u x^\kappa - 1)\right)^{\frac{1}{q-1}}dx, \quad C_u = \frac{C}{U}. \tag{12}$$

Both integrals in these equations may be calculated with the use of a tabulated [14] integral

$$I = \int_0^\infty \frac{x^{\mu-1}dx}{(a+bx^\nu)^\lambda} = \frac{1}{\nu a^\lambda}\left(\frac{a}{b}\right)^{\frac{\mu}{\nu}}\frac{\Gamma[\frac{\mu}{\nu}]\Gamma[\lambda-\frac{\mu}{\nu}]}{\Gamma[\lambda]} \tag{13}$$

under condition of convergence

$$0 < \frac{\mu}{\nu} < \lambda, \quad (\lambda > 1). \tag{14}$$

For the integrals in Eqs. (16) and (17) this condition becomes

$$0 < \frac{1+\kappa}{\kappa} < \frac{1}{1-q}. \tag{15}$$

Then, finally, we find from Eqs. (11), (12) with the use of the relation $\Gamma[1+z] = z\Gamma[z]$, that

$$\beta U = \frac{1}{\kappa} \text{ for all } q. \tag{16}$$

Independence of this relation on q means that it is true, in particular, for the limit case $q = 1$ where the Gibbs distribution take a place and therefore

$$\beta = \beta_0 \equiv 1/k_BT \text{ for all } q. \tag{17}$$

at least for power-law Hamiltonian.

When $H = p^2/2m$ (that is, $\kappa = 2$) we get from (16) and (17) that $U = \frac{1}{2}k_B T$ as it should be waited for one-dimensional ideal gas.

Besides, the Lagrange parameter β can be eliminated from the Renyi distribution (7) with the use of Eq. (16) and we have, alternatively,

$$p^R(x|q,\kappa) = Z^{-1}\left(1 - \frac{q-1}{\kappa q}(C_u x^\kappa - 1)\right)^{\frac{1}{q-1}}, \quad H(x) = Cx^\kappa \tag{18}$$

or

$$p_i^R(q,\kappa) = Z^{-1}\left(1 - \frac{q-1}{\kappa q}(C_u x_i^\kappa - 1)\right)^{\frac{1}{q-1}}, \quad H_i = Cx_i^\kappa. \tag{19}$$

So, at least for power-law Hamiltonian, the Lagrange multiplier β does not depend on the Renyi parameter q and coincides with the Gibbs parameter $\beta_0 = 1/k_B T$, and, moreover, can be excluded at all with the use of the relation (16).

The problem to be solved for an unique definition of the Renyi distribution is determination of a value of the Renyi parameter q . This will be the subject of the next section.

3 The most probable value of the Renyi parameter

An excellent example of a physical non-Gibbsian system was pointed by Wilk and Wlodarchuk [15]. They took into consideration fluctuations of both energy and temperature of a minor part of a large equilibrium system. This is a radical difference of their approach from the traditional Gibbs method in which temperature is a constant characterizing the thermostat. As a result, their approach leads [16] to the Renyi distribution with the parameter q expressed via heat capacity C_V of the minor subsystem

$$q = \frac{C_V - k_B}{C_V}.$$

The approach by Wilk and Wlodarchuk was advanced by Beck and Cohen [17, 18] who coined it the new term "superstatistics". In the frame of superstatistics, the parameter q is defined by physical properties of a system.

On the other side, there are many stochastic systems for which we have no information related to a source of fluctuations. In that cases the parameter q can not be determined with the use of the superstatistics.

Here, a useful extension of MEP is proposed. It consists in looking for a maximum of the Renyi or Boltzmann entrpopies in space of the Renyi distributions with different values of q .

It have appeared that both $S_R[p^R(x|q,\kappa)]$ and $S_B[p^R(x|q,\kappa)]$ attain their maximums at boundaries of the the range of possible values of q defined by inequalities (15) (see Fig. 1). Moreover, the Boltzmann entropy attains its maximum value at $q = 1$, that corresponds to the Gibbs distribution.

On contrary, the Renyi entropy attains its maximum value at the minimal possible value of q which satisfy the inequality (15), that is,

$$q_{min} = \frac{1}{1+\kappa}. \tag{20}$$

For $q < q_{min}$, the integral (10) diverges and, therefore, the Renyi distribution does not determine the average value $U = \langle H \rangle_p$, that is a violation of the condition of MEP.

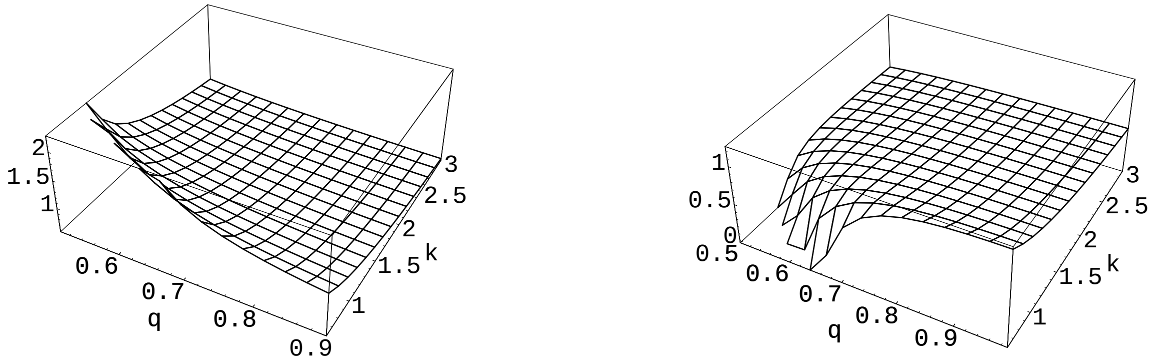

Figure 1: The entropies $S_R[p^R(x|q,\kappa)]/k_B$ (left) and $S_B[p^R(x|q,\kappa)]/k_B$ (right) for power–law Hamiltonian with the exponent κ within the range $q > 1/(1+\kappa)$.

Thus, it is found that the maximum of the Renyi entropy is realized at $q = q_{min}$ and it is just the value of the Renyi parameter that should be used for the discussed particular case of the power-law Hamiltonian if we have no additional information on behaviour of the stochastic process under consideration.

Substitution of $q = q_{min}$ into Eq. (22) leads to

$$p \sim x^{-(\kappa+1)} \tag{21}$$

Thus, for $q = q_{min}$ the Renyi distribution for a system with the power–law Hamiltonian becomes PLD over the whole range of x .

For a particular case of the impact fragmentation where $H \sim m^{2/3}$ the power-law distribution of fragments over their masses m follows from (21) as $p(m) \sim m^{5/3}$ that coincides with results of our previous analysis [10] and experimental observations [9].

For another particular case $\kappa = 1$ PLD is $p \sim x^2$. Such form of the Zipf-Pareto law is the most useful in social, biological and humanitarian sciences. Just the same exponent of PLD was demonstrated [19] in cosmic ray physics for energy spectra of particles from atmospheric cascades.

It is necessary to notice here that inequalities (15) suggest in fact $q > q_{min}$, that is, $q = q_{min} + \epsilon$ where ϵ is a positive infinitesimal value. It is clear that $\epsilon(\ll 1)$ should be a finite constant in real physical systems. Accounting for ϵ gives rise to the Renyi distribution in the form

$$p^R(x) = Z^{-1}(C_u x)^{-(\kappa+1)(1+\epsilon\frac{\kappa+1}{\kappa})}\left[1 - \epsilon(\kappa+1)^2(1 - C_u x^{-\kappa})\right]^{-\frac{\kappa+1}{\kappa}(1+\epsilon\frac{\kappa+1}{\kappa})} \tag{22}$$

For sufficiently great x 's this Renyi distribution passes to PLD where all terms with ϵ can be neglected.

On the other side, for small $x \ll 1$, only the term $\epsilon(\kappa+1)^2 C_u x^{-\kappa}$ may be accounted in the expression in the square brackets, so we get

$$p^R(x)|_{x\ll 1} \sim (\epsilon(\kappa+1)^2)^{-\frac{\kappa+1}{\kappa}} \tag{23}$$

This equation points to the fact that the asymptote to the Renyi distribution for small $x's$ is a constant of which value is determined by $\epsilon \ll 1$.

The picture of the Renyi distribution over the whole range of x is illustrated in the Fig. 2 for particular cases of $\epsilon = 10^{-6}$, $\epsilon = 10^{-5}$ and $\epsilon = 10^{-4}$.

Now there is no methods for an unique theoretical determination of ϵ , so it may be considered as a free parameter. It can be estimated for those experimental data where the head

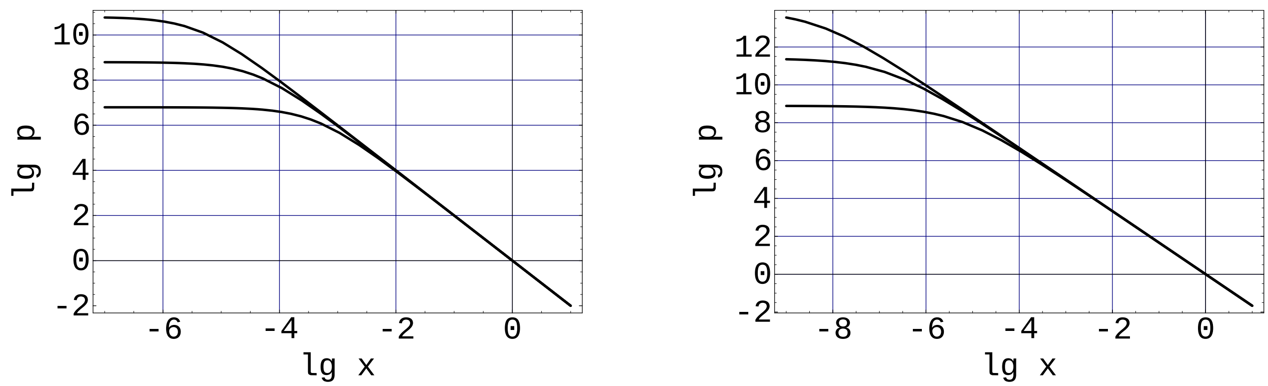

Figure 2: The Renyi distributions (non-normalized) for the power–law Hamiltonian $H \sim x^{\kappa}$, $\kappa = 1$ (left) and $\kappa = 2/3$ (right) and different values $\epsilon = 10^{-6}, 10^{-5}, 10^{-4}$ from upper to below in each graph.

part preceded PLD is presented. As an example, for the probability distribution of connections in WWW network [20] where the exponent of PLD is equal to -2.5, the parameter ϵ is estimated as $\sim 10^{-4}$.

4 Conclusions

Below, some results obtained in the present effort are summarized briefly. Maximum Entropy Principle applied to the Renyi entropy gives rise to a Renyi distribution that depends on the Renyi parameter q and two Lagrange multipliers α and β. The multiplier α corresponds to the condition of normalization of the distribution and may be eliminated with ease. The second Lagrange multiplier β corresponds to the condition of a fixed average energy $U = \langle H \rangle_p$ just as $\beta_0 = 1/k_B T$ in the Gibbs distribution function. It should be noted here that the connection of β_0 with the thermodynamic temperature obviates the necessity to eliminate the second Lagrange multiplier from the Gibbs distribution. It is not so for the Lagrange multiplier β, at least up to a time when the new Renyi thermostatistics will be constructed and β will obtain a physical meaning.

It is shown here that for the particular case of a power–law Hamiltonian $H_i = C x^{\kappa}$ the Lagrange multiplier β does not depend on the Renyi parameter q and coincides with β_0. Moreover, it can be expressed in terms of U and κ and thus excluded at all from the Renyi distribution function.

In the absence of any additional information on a nature of the stochastic process, the q-parameter of the corresponding Renyi entropy is determined with the further use of MEP in the space of q-dependent Renyi distributions. When applying such MEP to the Boltzmann entropy the maximum is found at $q = 1$ that corresponds to the Gibbs distribution. Maximum of the Renyi entropy is found at $q = 1/(1+\kappa)$. The exponent of the power–law distribution for such q is $-(1+\kappa)$ that agrees with observable data for stochastic systems with the power–law Hamiltonians.

It should be noticed that all above estimations of the parameters of the Renyi distribution (7) and the exponent of PLD are true as well for the escort version of the Tsallis' distribution [12]

$$p_i^{(Ts)} = Z_{Ts}^{-1} \left(1 - \beta^* (1 - q') \Delta H_i\right)^{\frac{q'}{1-q'}} \tag{24}$$

because of both distributions are identical if $q' = 1/q$. Really in this case

$$1 - q' = \frac{q-1}{q}, \quad \frac{q'}{1-q'} = \frac{1}{q-1} \tag{25}$$

and β^* is determined by the same second additional condition of MEP as well as β .

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